Graded sheaves and Applications

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Summary

1. Motivation: categorification of Hecke algebra
2. Graded sheaves on Artin stacks
3. Applications (GNR conjecture, Serre duality)
4. Expectations and Conjectures

Hecke algebra

- Symmetric group S_n is generated by $\sigma_1, ...\sigma_{n-1}$, subject to the relations:
  1. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
  2. $\sigma_i \sigma_j = \sigma_j \sigma_i, \ \ \forall |i-j| > 1$
  3. $\sigma_i^2 = 1$

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- $\mathbb{Z}[S_n]$ has a 1-parameter deformation:
  H_n is a $\mathbb{Z}[q, q^{-1}]$-algebra generated by $T_1, ..., T_{n-1}$ subject to the relation:
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  2. $T_iT_j = T_jT_i, \quad \forall|i-j| > 1$
  3. $T_i^2 = (q-1)T_i + q$

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- Hecke algebra H_W is defined similarly for general Weyl group W.

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Categorification of Hecke algebra using Tate sheaves

Theorem (Soergel, Beilinson–Ginzburg–Soergel)

Let $\mathcal{H}_G := D^b_{Tate}(B_0\backslash G_0/B_0) \subseteq D^b(B_0\backslash G_0/B_0)$ be the subcategory "generated" by $IC_w\langle k\rangle, w \in W$. Then

$$\mathcal{K}(\mathcal{H}_G) \simeq H_W$$

$$[IC_w\langle k\rangle] \mapsto q^k KL_w$$

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Main Goal

Define a sheaf theory $\mathcal{S} : Stk \to \mathrm{Cat}^{st}_{\infty}$, with six functors formalism, such that $\mathcal{S}(B\backslash G/B) = \mathcal{H}_G$.

Mixed sheaves

Previous constructions has an intermediate step:

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For $X_0 = \mathrm{pt}_0$, the above inclusion essentially comes from

$$\sqrt{q}^{\mathbb{Z}} \hookrightarrow \Omega \hookrightarrow \mathbb{C}^\times$$

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$$D^b_{gr}(pt) := \mathrm{Vect}^{\mathbb{Z}} \twoheadleftarrow D^b_{mix}(pt_0) \hookrightarrow D^b(pt_0)$$

The arrow $\twoheadleftarrow$ sending $(V, \varphi) \mapsto \oplus_i V_i$, where $V_i \subseteq V$ is the subspace of Frobenius weight i.

Definition of graded sheaves

Definition(Ho–L.)

Let $X/\overline{\mathbb{F}}_q$ be a finite type Artin stack, X_0 be a rational form of X.

$$D^b_{gr}(X) := D^b_{mix}(X_0) \otimes_{D^b_{mix}(pt_0)} D^b_{\mathrm{gr}}(pt)$$

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$D^b_{mix}(X_0)$ is equipped with

- full subcategories $(D^{w\leq 0}_{mix}(X_0), D^{w\geq 0}_{mix}(X_0))$ by Frobenius weights
- perverse t-structure $(D^{t\leq 0}_{mix}(X_0), D^{t\geq 0}_{mix}(X_0))$

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Theorem (Ho–L.)

$D^b_{gr}(X)$ has a weight structure and a perverse t-structure, compatible with the ones from $D^b_{mix}(X_0)$ and $D^b(X)$. Moreover, the weight structure and t-structure are transverse ($\Rightarrow$ any pure object $D^b_{gr}(X)$ is semisimple).

t-structure vs. weight structure

t-structure $(D^{t\leq 0}, D^{t\geq 0})$	**Weight structure** $(D^{w\leq 0}, D^{w\geq 0})$
Axioms:	**Axioms:**
(i) $D^{t\leq 0}[1] \subseteq D^{t\leq 0}$ $D^{t\geq 0}[-1] \subseteq D^{t\geq 0}$	(i) $D^{w\leq 0}[-1] \subseteq D^{w\leq 0}$ $D^{w\geq 0}[1] \subseteq D^{w\geq 0}$
(ii) $\mathrm{Hom}(c, d) = 0$ for $c \in D^{t\leq 0}$, $d \in D^{t\geq 1}$	(ii) Same
(iii) For any $c \in D$, $\exists$ triangle $c_{\leq 0} \to c \to c_{\geq 1}$	(iii) Same
Examples:	**Examples:**
$\mathcal{A}$ abelian category: $D^b(\mathcal{A})^{t\leq 0} = \{$ complexes whose cohomology in non-positive degrees$\}$	$\mathcal{B}$ additive category: $K^b(\mathcal{B})^{w\leq 0} = \{$ complexes in non-negative degrees$\}$

t-structure vs. weight structure con'd

t-structure $(D^{t\leq 0}, D^{t\geq 0})$	**Weight structure** $(D^{w\leq 0}, D^{w\geq 0})$
$\mathrm{Ext}^{<0}(c,d)=0$, for $c,d\in D^{t=0}$ $D^{t=0}$ is classical abelian category	$\mathrm{Ext}^{>0}(c,d)=0$, for $c,d\in D^{w=0}$ $D^{w=0}$ is additive ∞-category
$(D^{t\leq 0}_{mix}(-), D^{t\geq 0}_{mix}(-))$ is a t-structure.	$(D^{w\leq 0}_{mix}(-), D^{w\geq 0}_{mix}(-))$ is NOT a weight structure
$(D^{t\leq 0}_{gr}(-), D^{t\geq 0}_{gr}(-))$ is a t-structure.	$(D^{w\leq 0}_{gr}(-), D^{w\geq 0}_{gr}(-))$ is a weight structure!
Beilinson realization functor $real : D^b(D^{t=0}) \to D$	Bondarko weight complex functor $wt : D \to K^b(hD^{w=0})$
$D^{t\leq 0}(or\ D^{t\geq 0}) \hookrightarrow D$ has adjoint $\tau_{t\leq 0}, \tau_{t\geq 0}$ is canonical	No adjoint $\tau_{w\leq 0}, \tau_{w\geq 0}$ is not canonical
t,w are transverse $:= \tau_{w\leq i}, \tau_{w\geq i}$ are functorial and t-exact on $D^{t=0}$ transversality $\Rightarrow$ any $c\in D^{w=0}$ is semisimple for t-structure	
For any $F : (C,w)\to(D,t)$, define the composition $\widehat{F} : C \xrightarrow{wt} K^b(C^{w=0}) \xrightarrow{K^b(H^*\circ F)} \oplus_{\mathbb{Z}} K^b(D^{t=0}) \longrightarrow \oplus_{\mathbb{Z}} D^b(D^{t=0}) \xrightarrow{real} \oplus_{\mathbb{Z}} D$	

Functorial categorification of Hecke algebra

Proposition (Ho–L.)

Suppose $RHom_{D^b_{\mathrm{gr}}(X)}(\mathrm{IC}_1, \mathrm{IC}_2)$ is pure for all IC-sheaves $\mathrm{IC}_1, \mathrm{IC}_2$, then there is an canonical equivalence:

$$D^b_{\mathrm{gr}}(X) \simeq K^b(SS(X))$$

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Space X satisfies the purity condition:

- $B\backslash G/B$ (Soergel, categorification of H_W) $\Rightarrow D^b_{gr}(B\backslash G/B) = \mathcal{H}_G$
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Space X do NOT satisfy the purity condition: $\frac{G}{B}, \frac{G}{G}, \ldots$

Further properties/remarks

- Let $D^b_\infty(X) \subset D^b(X)$ be the category generated by sheaves comming from some rational form X_0.
- $D^b_{mix}(X_0) \to D^b_{gr}(X) \to D^b_\infty(X) \subset D^b(X)$, functors in general not essentially surjective.

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- Expect the Hodge counterpart

$$D^b_{gr,Hod}(X) = D^b(MHM)(X) \otimes_{D^b(MHS)} \mathrm{Vect}^{gr}$$

Applications of graded sheaves

Ocneanu, Jones: HOMFLY-PT polynomial of a braid β can be realized as a trace on Hecke algebra.

$$\beta \in Br_n \to H_n \to hh(H_n) \to \mathbb{C}[q, z]$$

Proposition (Webster–Williamson, Shende–Treumann–Zaslow, Ho–L.)

Let $G = GL_n$, the composition is the Khovanov–Rozansky homology $HHH(\beta)$ of the braid β:

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GNR conjecture: what are the object corepresenting summands of $\widehat{R\Gamma}$?
Restrict to (unipotent) character sheaves $Ch_{G,\mathrm{gr}} := \langle \mathrm{Im}(q_! p^*) \rangle \subseteq D^b_{gr}(\frac{G}{G})$.

Gorsky-Negut-Rasmussen Conjecture

Theorem (Ho–L.)

1. There is an equivalence of ∞-categories:

$$\Phi : Ch_{GL_n,\mathrm{gr}} \simeq \mathrm{Coh}^{per}_{\mathbb{C}^\times \times \mathbb{C}^\times}(\mathrm{Hilb}_n(\mathbb{C}^2))_{y=0}$$

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Remark: Similar argument shows that

$$H^*(\widetilde{\mathcal{M}(\beta)}) = RHom^{\mathbb{C}^\times \times \mathbb{C}^\times}_{\mathrm{Hilb}_n}(\mathcal{P}, \mathcal{F}_\beta)^{per},$$

for $\widetilde{\mathcal{M}(\beta)}$ = braid variety, and $\mathcal{P}$ = Procesi bundle.

Relative Serre duality on Hecke categories

Let $L \subset G$ Levi subgroup, $i : \mathcal{H}_L \hookrightarrow \mathcal{H}_G$, and i^L, i^R be left and right adjoints. Let $FT_G = \Delta_{w_0}^2$, for w_0 maximal elements. Put $FT_{G,L} = FT_L^{-1} \star FT_G$.

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Remark: This is in analogue with the relative Serre duality in algebraic geometry: for $p : X \to S$ a smooth proper map, then we have $(p^*)^R = (p^*)^L(\omega_{X/S}^{-1} \otimes -)$.
Our proof is inspired by Kapranov's result on (absolute) Serre duality in category $\mathcal{O}$.

Further expectations

Recall we have the equivalence

$$\widehat{Ch}_{GL_n,gr} \xrightarrow{\Phi} \mathrm{Coh}^{per}_{\mathbb{C}^\times \times \mathbb{C}^\times}(\mathrm{Hilb}_n(\mathbb{C}^2))$$

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Some evidences:

1. Φ takes (ind-)unit to (ind-)unit.
2. GNR conjecture part II: $\mathcal{F}_{\beta\star FT} \simeq \mathcal{F}_\beta \otimes \mathcal{O}(1)$. This would follow from the monoidal conjecture, rigidity of $\widehat{Ch}_{GL_n,gr}$ and $p_* q^! \Phi^{-1}(\mathcal{O}(1)) \simeq FT \in \mathcal{H}_G$ [Bezrukavnikov-Tolmochov].
3. Cohomology of character stack

Character sheaves as 2d TFT

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  - value on S^1 = class functions $Fun(\frac{\Gamma}{\Gamma})$ (a Frobenius algebra)
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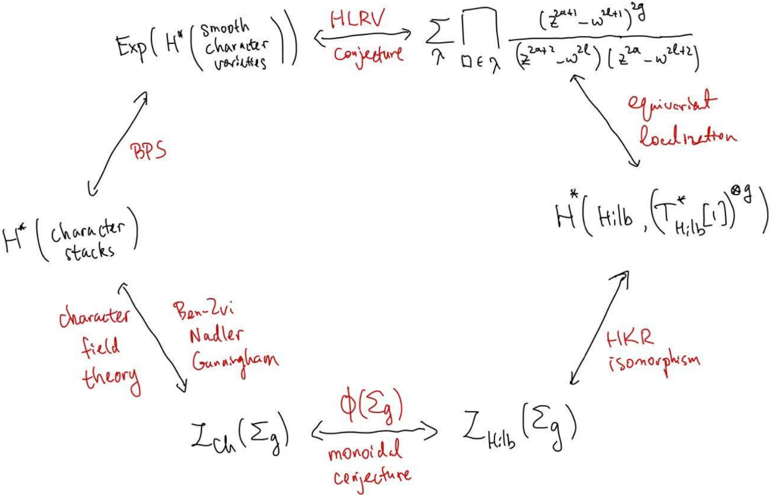
where the second equivalence is because (for X smooth):

$$\mathrm{Vect} \xrightarrow{\mathbf{1}=\mathcal{O}} \mathrm{Coh}(X) \xrightarrow{\Delta_* = \otimes^\vee} \mathrm{Coh}(X \times X) \xrightarrow{\Delta^* = \otimes} \mathrm{Coh}(X) \xrightarrow{\Delta_*} \ldots \xrightarrow{\Delta^*} \mathrm{Coh}(X) \xrightarrow{\mathbf{1}^\vee = H^*} \mathrm{Vect}$$

A commutative diagram?



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