Beijing-Shanghai Summer School in Mathematics, 2025, exercises for the lecture.

0.1. **Geometry of Springer fibers.** We write $\mathcal{B}$ for the flag space G/B, we also write $\mathcal{B}_n$ for $\mathcal{B}$ if $G = GL(n)$. When working with $GL(n)$ we write V for the n-dimensional space it acts on (the tautological representation).

(1) Let G be a reductive group. Fix a parabolic P and let nilpotent e be chosen generically in the radical of P. (A nilpotent orbit obtained this way is called Richardson. Every nilpotent orbit in $gl(n)$ or $sl(n)$ is Richardson but in general this is not so.)

Show that the set of Borel subalgebras contained in P is a component of the Springer fiber $\mathcal{B}_e$, this component is isomorphic to the flag space of the Levi L.

[You can assume that $G = GL(n)$ if you prefer.]

(2) (a) Let $e \in sl(n)$ be such that $e^2 = 0$, $rank(e) = r$. Check that e is generic in the group of block upper triangular matrices with zero diagonal blocks, where the blocks are of the sizes $(n-r)$ and $r = rank(e)$.

Thus the previous problem gives a component X of $\mathcal{B}_e$ isomorphic to $\mathcal{B}_{n-r} \times \mathcal{B}_r$. Describe another component X' of $\mathcal{B}_e$ isomorphic to a $\mathbb{P}^1$ bundle over $D \subset \mathcal{B}_{n-r} \times \mathcal{B}_r$ where the divisor D parametrizes pairs of flags $(V_0 \subset \cdots \subset V_{n-r}), (V_0' \subset \cdots \subset V_r')$ such that $V_1' \subset V_{n-r-1}$.

(b) For $i = 1, \ldots, n-1$ a line of type i in $\mathcal{B}_n$ is the set of flags $(0 = V_0 \subset \cdots \subset V_n = V)$ with fixed $V_1, \ldots, V_{i-1}; V_{i+1}, \ldots, V_n$.

Let X be a component of Springer fiber, we say that X admits a type i fibration (or i-fibers) if it is a union of lines of type i. Check the following.

If X admits a type i fibration for all i then $X = \mathcal{B}_n$ and $e = 0$. If i is such that X does not i-fiber, show that there exists a divisor $D \subset X$ and another component X' of $\mathcal{B}_e$ which is the union of all type i lines intersecting D.

If X does not admit a type i fibration for a unique i then $e^2 = 0$ and X is as in the previous part of the problem.

[This can also be generalized to any reductive group].

(c) For $r = 1$ show that $\mathcal{B}_e$ has $(n-1)$ components where the i-th component X_i fibers over $Gr(i-1, n-2)$ with fiber $\mathcal{B}_{i-1} \times \mathcal{B}_{n-i-1}$.

(3) Let $e \in sl(n)$ be the nilpotent with Jordan blocks of size $(1, n-1)$ (the subregular nilpotent).

Show that the Springer fiber is a union of $(n-1)$ projective lines, describe their intersection pattern and the explicit matrix by which a simple reflection $s_i = (i, i+1)$ acts on top homology in the basis of components.

Bonus problem: generalize this to $Sp(2n)$ and $SO(n)$ (in the latter case consider the case of odd and even n separately), i.e. find a nilpotent with a one dimensional Springer fiber and describe this fiber explicitly.

(4) The centralizer Z_e of e acts on $\mathcal{B}_e$. This problem gives an example where this action has infinitely many orbits.

Let $e \in sl(8)$ be the nilpotent with two equal Jordan blocks. Consider the set X of flags $(V_0 \subset \cdots \subset V_8)$ such that $V_2 = Ker(e)$, $V_4 = Ker(e^2)$, $V_6 = Ker(e^3)$, this is a Z_e-invariant closed subvariety in $\mathcal{B}_e$. Set $U =$

$Ker(e)$. Construct a Z_e-invariant onto map $\mathcal{B}_e \to \mathbb{P}(U)^4$. Conclude that the action of Z_e on X and hence on $\mathcal{B}_e$ has infinitely many orbits.

One can optimize this example as follows: let $e \in sl(6)$ of Jordan type $(2, 4)$ and let $H \subset V$ be the unique 5-dimensional e-invariant subspace such that $e|_H$ has Jordan type $(3, 2)$ (namely, $H = Im(e) + Ker(e^2)$). We let Y be the set of e-invariant flags such that $V_5 = H$, $V_3 = e(H)$, $V_1 = e^2(H)$, this is a Z_e-invariant closed subvariety in $\mathcal{B}_e$. Let $U' = H/eH$. Show that Z_e acts on $\mathbb{P}(U')$ fixing two points: the lines $Im(e)/eH$ and $Ker(e^2)/eH$. Construct a Z_e invariant onto map $Y \to \mathbb{P}(U')^2$, deduce that Z_e acts on Y, and hence on $\mathcal{B}_e$ with infinitely many orbits.

(5) (*) Let $e \in sl(n)$ be a nilpotent element, its Jordan type defines a Young diagram λ. Given a flag $(V_0 \subset \cdots \subset V_n = V)$ invariant under e one can consider the Jordan types of $e|_{V_i}$ for $i = 1, \dots, n$, this gives a nested collection of Young diagrams $\lambda_1 \subset \lambda_2 \subset \cdots \subset \lambda$, this collection corresponds to a standard tableau of shape λ: we place i at the unique box in λ_i which is not in λ_{i-1}. This defines a map from $\mathcal{B}_e$ to the set of standard tableaux of shape λ. Show that the preimage of every tableau is isomorphic to the affine space $\mathbb{A}^d$, $d = \dim(\mathcal{B}_e)$, the closure of this preimage is a component of $\mathcal{B}_e$, and this defines a bijection between the set of components and the set of standard tableaux of shape λ.

0.2. **Character sheaves.** The next few problems give basic examples of calculations with character sheaves.

(6) Let A be a connected abelian algebraic group over $\mathbb{F}_q$ (for example, the additive group $\mathbb{G}_a$ or the multiplicative group $\mathbb{G}_m$). The Artin-Schreier homomorphism $\alpha : A \to A$ is given by $\alpha = Fr - Id$ where Fr is the Frobenius morphism. Thus α is an etale covering with Galois group $A(\mathbb{F}_q)$, so $A(\mathbb{F}_q)$ acts on the direct image of the constant sheaf under α, inducing a decomposition as a direct sum indexed by characters of $A(\mathbb{F}_q)$. For a character χ of $A(\mathbb{F}_q)$ let $\mathcal{F}_\chi$ be the corresponding summand. Show that the trace of Frobenius function $f_{\mathcal{F}_\chi}$ equals χ.

(7) Consider the set U_{reg} of regular unipotent elements in $G = SL(n)$. Describe order n rank one local systems on U_{reg} and compute their trace of Frobenius functions.

[One can show that extending such a local system by zero to G one gets an irreducible perverse sheaf, in fact, a character sheaf].

(8) Let $G = GL(n)$. The principal series irreducible representations of $G(\mathbb{F}_q)$ (i.e. representations appearing in $\mathbb{C}[G/B]$) are in bijection with irreducible representations of the symmetric group. For an irr. rep ρ of S_n the character of the corresponding irr rep R_ρ of $G(\mathbb{F}_q)$ is the trace of Frobenius on the ρ-isotypic component of the Grothendieck-Springer sheaf (the direct image of the constant sheaf under the Grothendieck-Springer map $\tilde{G} \to G$ where $\tilde{G} \subset G \times \mathcal{B}$ is given by $\tilde{G} = \{(g, x) \mid g(x) = x\}$.)

Use this to compute explicitly dimensions of all principal series irr. reps of $GL(n)$ for $n \leq 4$. For any n describe the character of R_{sgn} and $R_{\tau\otimes sgn}$ restricted to the set of unipotent elements; here sgn is the sign character and τ is the standard reflection representation of S_n.

[You can use that for $G = GL(n)$ the restriction map $H^*(\mathcal{B}) \to H^*(\mathcal{B}_e)$ is onto for all e].

(9) Let G be a reductive group and $T \subset G$ a maximal torus, recall that conjugacy class of T corresponds to a conjugacy class in the Weyl group $w_T \in W$. For a character θ of $T(\mathbb{F}_q)$ Deligne and Lustztig defined a virtual representation $R_{T,\theta}$ of G (it is an irreducible representation if θ is generic). The character value of $R_{T,\theta}$ at a unipotent element $u \in G$ does not depend on θ, it equals $Tr([q] \circ w, H^*(\mathcal{B}_u))$, where $[q]$ is the operator acting by q^i on $H^{2i}(\mathcal{B}_u)$ and w is acting via the Springer representation.

Use this to compute dimensions of all DL representations of $GL(n)$ for $n \leq 4$, as well as their character values at a regular and subregular unipotent element u.